A Generic Analytical Model of Fractance Response in Time Domain

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***Abstract*—In this research, a generic analytical model of voltage response of fractance in time domain which is often cited in various disciplines such as control engineering and electronic engineering etc., has been proposed. This model can be applied to fractance of any order under any type of current excitation. So, it has been found to be beneficial to various fractance involved disciplines stated above which are the crucial basis in the development of technology for internet of things due to its generality.**

Keywords—fractance; fractional order calculus; Fourier series; generalized trigonometric functions;internet of things

I. INTRODUCTION

Fractance or fractional impedance is the impedance which its order (α) can be fractional i.e. $0 \leq \alpha \leq 1$ [1-3]. It is often cited in various disciplines which serve as the foundation in the development of technology for internet of things e.g. control engineering [4] and electronic engineering [5] etc. In many circumstances, various characteristics of fractance are needed to be precisely determined [6-9]. Traditionally, they can be determined by using the numerical simulation in time domain based on the measured voltage data obtained from exciting the fractance with the predetermined current waveform [10]. This traditional methodology is cumbersome and yields non analytical results which are imprecise compared to the analytical ones. It can be seen that much of the effort can be reduced and the precise analytical result can be expected if the time domain responses of the fractance can be analytically obtained. By this motivation, the analytical expressions of these responses have been proposed in previous studies such as [3], [11] and [12] etc. Unfortunately, only specific current excitations have been concerned in these previous works.

Hence, a generic analytical model of voltage response of fractance in time domain has been proposed in this research. This model can be applied to fractance of any α under the current excitation of any type. Such generality has been obtained from the usage of Fourier series as the basis of the model. So, this model has been found to be beneficial various aforementioned disciplines which are the foundation in the development of technology for internet of things and the fractance is of interested.

II. THE PROPOSED MODEL

In this section, the proposed model will be presented by starting from its derivation. Mathematically, any exciting current ($I(t)$) with arbitrary period, T can be expressed in term of its Fourier series [13] as

$$I(t) = I_0 + \sum_{n=1}^{\infty}[I_{an}\cos(n\omega t) + I_{bn}\sin(n\omega t)] \tag{1}$$

where

$$I_0 = \frac{1}{T}\int_{-T/2}^{T/2} I(t)dt \tag{1a}$$

$$I_{an} = \frac{1}{T}\int_{-T/2}^{T/2} I(t)\cos(n\omega t)dt \tag{1b}$$

$$I_{bn} = \frac{1}{T}\int_{-T/2}^{T/2} I(t)\sin(n\omega t)dt \tag{1c}$$

For any fractance with magnitude K and order α which its impedance function is $Z(s) = Ks^{\alpha}$, its s-domain voltage response ($V(s)$) can be given in term of its s-domain exciting current ($I(s)$) as follows

$$V(s) = Ks^{\alpha} I(s) \tag{2}$$

So, the time domain voltage response ($V(t)$) can be given by taking the inverse Laplace transform to $V(s)$ which yields

$$V(t) = KD^{\alpha}[I(t)] \tag{3}$$

where $D^{\alpha}[\ ]$ denotes the fractional order derivative operator.

Among various mathematical definitions of fractional order derivative [14-16] etc., the Riemann-Liouvielle definition has been chosen for this research. Since $0 \leq \alpha \leq 1$, the Riemann-Liouvielle definition with $m = 1$ has been adopted for defining $D^{\alpha}[\ ]$. As a result, (3) become

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$$V(t) = \frac{K}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} I(\tau) d\tau \quad (4)$$

By using (1) and (4), $V(t)$ can be found as

$$V(t) = K[D^{\alpha}[I_0] + \sum_{n=1}^{\infty} [I_{an} D^{\alpha}[\cos(n\omega t)] + I_{bn} D^{\alpha}[\sin(n\omega t)]]] \quad (5)$$

After performing the fractional order differentiation, the proposed analytical model can be finally obtained in term of $V(t)$ as follows

$$V(t) = K[\frac{I_0 t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}[(n\omega)^{\alpha}[I_{an}[\cos_{\alpha}(n\omega t)\cos(\frac{\alpha\pi}{2}) - \sin_{\alpha}(n\omega t)\sin(\frac{\alpha\pi}{2})] + I_{bn}[\sin_{\alpha}(n\omega t)\cos(\frac{\alpha\pi}{2}) + \cos_{\alpha}(n\omega t)\sin(\frac{\alpha\pi}{2})]]]] \quad (6)$$

where I_0, I_{an} and I_{bn} can be given by (1a)-(1c), moreover, $\sin_{\alpha}(\ )$ and $\cos_{\alpha}(\ )$ denote the generalized sine and cosine of order α respectively. For any variable x, these generalized trigonometric functions can be defined in term of the normal ones as

$$\sin_{\alpha}(x) = \sum_{k=0}^{\infty} [\frac{x^{k-\alpha} \sin(0.5(k-\alpha)\pi)}{\Gamma(k-\alpha+1)}] \quad (7)$$

$$\cos_{\alpha}(x) = \sum_{k=0}^{\infty} [\frac{x^{k-\alpha} \cos(0.5(k-\alpha)\pi)}{\Gamma(k-\alpha+1)}] \quad (8)$$

By using the proposed model, $V(t)$ due to any $I(t)$ can be analytically obtained as will be shown in the subsequent section. So, the proposed model has been found to be beneficial to the study of any fractance involved systems. Before leaving this section, it should be mentioned here that since the generalized trigonometric functions are asymptotically converged to normal ones and the first term of $V(t)$ is asymptotically very small, $V(t)$ is asymptotically $V_{asymp}(t)$ which can be approximately given by

$$V_{asymp}(t) = K\sum_{n=1}^{\infty}[I_{an}\cos(n\omega t + \frac{\alpha\pi}{2}) + I_{bn}\sin(n\omega t + \frac{\alpha\pi}{2})] \quad (9)$$

On the other hand, $V(t)$ at transient ($V_{trans}(t)$) which is asymptotically vanished, can be analytically obtained as follows.

$$V_{trans}(t) = K[\frac{I_0 t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}[(n\omega)^{\alpha}[I_{an}[(\cos_{\alpha}(n\omega t) - \cos(n\omega t))\cos(\frac{\alpha\pi}{2}) - (\sin_{\alpha}(n\omega t) - \sin(n\omega t))\sin(\frac{\alpha\pi}{2})] + I_{bn}[(\sin_{\alpha}(n\omega t) - \sin(n\omega t))\cos(\frac{\alpha\pi}{2}) + (\cos_{\alpha}(n\omega t) - \cos(n\omega t))\sin(\frac{\alpha\pi}{2})]]]] \quad (10)$$

III. ITS APPLICATION

In this section, the application of the proposed model i.e. analytically obtaining $V(t)$'s due to specific $I(t)$'s will be shown. As an exmple, $I(t) = |I_0\sin(t)|$ which is a mathematical expression of a rectified sinusoidal waveform with $T = 2\pi$ and zero phase shift, will be considered. With this $I(t)$, $V(t)$ for fractance with magnitude K and order α can be obtained by applying the proposed model as follows

$$V(t) = \frac{4KI_0}{\pi}[\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}[\frac{n^{\alpha}}{1-n^2}[\cos_{\alpha}(nt)\cos(\frac{\alpha\pi}{2}) - \sin_{\alpha}(nt)\sin(\frac{\alpha\pi}{2})]]] \quad (11)$$

It can be seen that this $V(t)$ can be alternatively given in term of a periodic waveform with $T = 2\pi$ and the following analytical expression for $0 \le t \le 2\pi$.

$$V(t) = \left| KI_0[\cos_{\alpha}(t)\sin(\frac{\alpha\pi}{2}) + \sin_{\alpha}(t)\cos(\frac{\alpha\pi}{2})] \right| \quad (12)$$

Asymptotically, this $V(t)$ become $V_{\mathrm{asymp}}(t)$ which can be given by (13) and can be plotted against t and α by assuming that $I_0 = 1$ and $K = 1$ as shown in Fig.1. Furthermore, $V_{\mathrm{trans}}(t)$ can be analytically given by using (11) and (13) as in (14).

$$V_{asymp}(t) = \frac{4KI_0}{\pi}\sum_{n=1}^{\infty}[\frac{n^{\alpha}}{1-n^2}\cos(nt + \frac{\alpha\pi}{2})] \quad (13)$$

$$V_{trans}(t) = \frac{4KI_0}{\pi}[\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}[\frac{n^{\alpha}}{1-n^2}[(\cos_{\alpha}(nt) - \cos(nt))\cos(\frac{\alpha\pi}{2}) - (\sin_{\alpha}(nt) - \sin(nt))\sin(\frac{\alpha\pi}{2})]]] \quad (14)$$

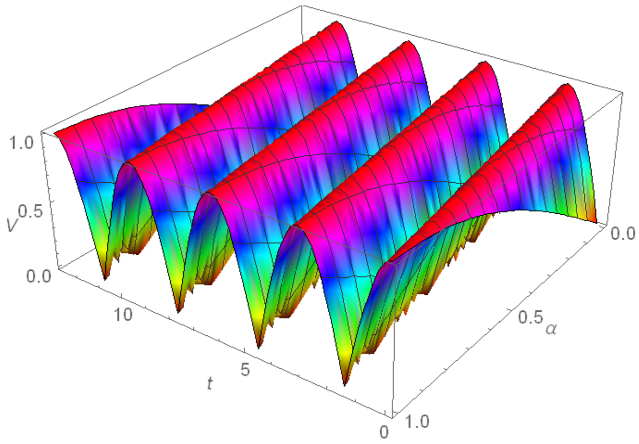

Fig.1. $V_{asymp}(t)$ due to $I(t) = |I_0\sin(t)|$ of fractance when $I_0 = 1$ and $K = 1$

As another example, $I(t)$ as a sawtooth waveform with $T = 2\pi$ will be focused. This $I(t)$ can be analytically given for $0 \leq t \leq 2\pi$ as $I(t) = I_0 t$. By using the proposed model, $V(t)$ for fractance with arbitrary values of K and α can be found as follows

$$V(t) = 2KI_0[\frac{\pi t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{n=1}^{\infty}[n^{\alpha-1}[\sin_\alpha(nt)\cos(\frac{\alpha\pi}{2}) + \cos_\alpha(nt)\sin(\frac{\alpha\pi}{2})]]] \tag{15}$$

Of course, $V_{trans}(t)$ and $V_{asymp}(t)$ for this case can be simply found respectively as

$$V_{trans}(t) = 2KI_0[\frac{\pi t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{n=1}^{\infty}[n^{\alpha-1}[(\sin_\alpha(nt) - \sin(nt))\cos(\frac{\alpha\pi}{2}) + (\cos_\alpha(nt) - \cos(nt))\sin(\frac{\alpha\pi}{2})]]]] \tag{16}$$

$$V_{asymp}(t) = -2KI_0\sum_{n=1}^{\infty}[n^{\alpha-1}\sin(nt + \frac{\alpha\pi}{2})] \tag{17}$$

If the fractances of our interested are a half order differentiator/inductor (α = 0.5) and a normal one (α = 0.1), their $V(t)$ can be given for $0 \leq t \leq 2\pi$ by (18a) and (18b) respectively. It can be seen from (18b) that the fractance acts as a theoretical integrator when α = 1 as expected.

$$V(t) = \frac{2KI_0}{\Gamma(0.5)}\sqrt{t} \tag{18a}$$

$$V(t) = KI_0 \tag{18b}$$

IV. Conclusion

A generic analytical model of voltage response of fractance of any α under any type of current excitation has been proposed. The Fourier series has been adopted as the basis of the model. This model has been shown to be applicable to fractances under various type of current excitations e.g. rectified sinusoidal and sawtooth wave etc. Furthermore, the voltage response at asymptotic and transient can also be simply determined by using this model. Hence, it has been found to be beneficial to various disciplines e.g. control engineering and electronic engineering etc., which are important to the development of technology for internet of things and the fractance is of interested.

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